Equilibrium and Stability of a Three-Dimensional Toroidal MHD Configuration Near its Magnetic Axis

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The expansion of a three-dimensional toroidal magnetohydrostatic equilibrium around its magnetic axis is reconsidered. Equilibrium and stability plasma-β estimates are obtained in connection with a discussion of stagnation points occurring in the third-order flux surfaces. The stability criteria entering the β-estimates are: (i) a necessary criterion for localized disturbances, (ii) a new sufficient criterion for configurations without longitudinal current. Hamada coordinates are used to evaluate these criteria.

I. Introduction

The equilibrium and stability of toroidal three-dimensional magnetohydrostatic configurations have already been treated several times [1–4]. The common basis of these investigations is the expansion of the equilibrium with respect to the distance from the magnetic axis; results are obtained from the third order of this expansion. Although this method has not only been developed generally but has also led to explicit results for specific configurations, it was found necessary for several reasons to reconsider this problem. The results in the literature suffer from special and ad hoc choices for the third-order coefficients occurring in the description of the geometry of flux surfaces. These third-order coefficients determine the existence and position of stagnation points, which in turn determine the aspect ratio of the plasma column. Therefore, a complete discussion of the stagnation points occurring in the third-order flux surfaces, which is also missing in the literature, is essential in order to obtain estimates for limiting equilibrium and stability β values. In addition, most of the final results are obtained for a small ellipticity parameter of the flux surfaces. This minor deficiency of the published results, which impedes their applicability to actual configurations, can also be eliminated.

In the course of the work it was found that it was essential to redevelop the theory self-sufficiently. This made it possible to check the algebraically tedious calculations step by step with the help of the Reduce [5] algebraic programming system. By this procedure we obtained a complete set of general formulae. In Sec. II we introduce the notation and obtain a general expression for the magnetic well, which is the most complicated quantity occurring in the stability criteria; in Sec. III we list the relevant equilibrium formulae; Sec. IV contains the stability criteria used; in Sec. V we discuss the stagnation points occurring in the third-order flux surfaces and their relation to the β estimates obtained from the stability criteria.

II. Notation, Description of Magnetic Surfaces, and Formal Expression for the Magnetic Well

Let the magnetic axis be described by its curvature $\varkappa$ and its torsion τ. If l describes the arc length along the magnetic axis and ϱ, φ are polar coordinates in a plane perpendicular to the magnetic axis, φ being counted from the normal of the magnetic axis in such a way that $\{\varrho, \varphi, l\}$ is a right-handed coordinate system, then its metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \varrho^2\,\mathrm{d}\varphi^2 - 2\,\tau\,\varrho^2\,\mathrm{d}\varphi\,\mathrm{d}l + [(1-\varkappa\,\varrho\cos\varphi)^2 + \tau^2\varrho^2]\,\mathrm{d}l^2\,,$$

$$\sqrt{g} = \varrho\,(1-\varkappa\,\varrho\cos\varphi)\,, \qquad (1)$$

where τ is defined to be positive for a magnetic axis whose moving trihedral turns counterclockwise. Furthermore, let the contravariant components B^ϱ, B^φ, B^l of the magnetic field $\mathbf{B}$ be given by

$$B^\varrho = a_1\,\varrho + a_2\,\varrho^2 + O(\varrho^3)\,,$$
$$B^\varphi = b_0 + b_1\,\varrho + O(\varrho^2)\,, \qquad (2)$$
$$B^l = c_0 + c_1\,\varrho + c_2\,\varrho^2 + c_3\,\varrho^3 + O(\varrho^4)\,,$$

let V be the volume inside a magnetic surface,

$$V = V_2\,\varrho^2 + V_3\,\varrho^3 + O(\varrho^4)\,,$$

and let Φ be the longitudinal flux.

Because of

$$V = \int \sqrt{g}\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,\mathrm{d}l\,, \quad \Phi = \int \sqrt{g}\,B^l\,\mathrm{d}\varrho\,\mathrm{d}\varphi \qquad (3)$$

one obtains

$$\Phi = c_0 F + O(\varrho^4) , \quad F = \int \sqrt{g}\, \mathrm{d}\varrho\, \mathrm{d}\varphi ,$$

where F is the cross-sectional area of the magnetic surfaces normal to the magnetic axis, so that

$$V = c_0 F \oint \mathrm{d}l/c_0 + 0(\varrho^4) . \tag{4}$$

We choose the following third-order representation for the area F

$$F = \pi \left[e(\bar{x}\cos\alpha - \bar{y}\sin\alpha)^2 + \frac{1}{e}(\bar{x}\sin\alpha + \bar{y}\cos\alpha)^2 - 4\varrho^3 \frac{\pi}{L}(\delta\cos u \sin^2 u + \Delta\cos^2 u \sin u)\right] , \tag{5}$$

with

$$u = \varphi + \alpha ,$$

$$\bar{x} = \varrho\cos\varphi + \frac{2\pi}{L}\frac{S}{e}\varrho^2\left(e\cos^2 u + \frac{1}{e}\sin^2 u\right) ,$$

$$\bar{y} = \varrho\sin\varphi + \frac{2\pi}{L}\frac{s}{e}\varrho^2\left(e\cos^2 u + \frac{1}{e}\sin^2 u\right) ,$$

where L is the length of the magnetic axis.

Then, e is the half-axis ratio of the elliptical (in second order) plasma cross-section. For $e<1\,(>1)$, α is the angle between the (bi-) normal of the magnetic axis and the major half-axis, and α is counted in such a way that it increases for cross-sections rotating counterclockwise with increasing arc length. Since the elliptical cross-section has to make $n/2$ (n integer) full turns over the length of the magnetic axis, $\alpha(L) - \alpha(0) = n\pi$. The non-dimensional third-order quantities δ and Δ describe symmetric and antisymmetric triangular deformations of the surface, where the symmetry holds with respect to $u=0$. The quantities S and s describe shifts of the magnetic surfaces with respect to the magnetic axis in the directions normal and binormal to the magnetic axis, S and s being positive for shifts antiparallel to the normal and the binormal respectively. We now express the quantity $\ddot{\Phi}/\dot{\Phi}$ characterizing the magnetic well on the magnetic axis (where the dot represents the derivative with respect to the volume V) in terms of the quantities introduced above. Because of Eqs. (3) a simple calculation leads to

$$\ddot{\Phi}/\dot{\Phi} = \int (1/c_0 V_2^3)\left[\tfrac{1}{2} V_2(c_2 - \varkappa c_1\cos\varphi) - V_3 c_1\right] \mathrm{d}\varphi\, \mathrm{d}l .$$

Anticipating a simple result of the equilibrium calculation (Sec. III), $c_1 = 2\varkappa c_0\cos\varphi$, one obtains

$$\ddot{\Phi}/\dot{\Phi} = \int \frac{\mathrm{d}l\,\mathrm{d}\varphi}{V_2^2}\left(\frac{c_2}{2c_0} - \varkappa^2\cos^2\varphi\right) - \frac{2}{L(\oint \mathrm{d}l/c_0)^2}\int \frac{\varkappa}{c_0^2} \cdot \left[\frac{4}{e}S - (\Delta\sin\alpha + \delta\cos\alpha)\right]\mathrm{d}l . \tag{6}$$

Here, apart from c_2, the three third-order coefficients δ, Δ and S occur. Since the equilibrium equations connect δ, Δ, S, and s, Eq. (6) does not describe $\ddot{\Phi}/\dot{\Phi}$ in terms of prescribable parameters.

III. Equilibrium Calculation

The equilibrium equations

$$\nabla\cdot\mathbf{B} = 0 , \quad \mathbf{j}\times\mathbf{B} = \nabla p , \quad \mathbf{j} = \nabla\times\mathbf{B}$$

are solved in the following way. The divergence equation

$$(\sqrt{g}B^\varrho)_{,\varrho} + (\sqrt{g}B^\varphi)_{,\varphi} + (\sqrt{g}B^l)_{,l} = 0 \tag{7 a}$$

and the two components of the momentum balance

$$\dot{p}\, V_{,\varrho} = \sqrt{g}(j^\varphi B^l - j^l B^\varphi) , \tag{7 b}$$

$$\dot{p}\, V_{,\varphi} = \sqrt{g}(j^l B^\varrho - j^\varrho B^l) , \tag{7 c}$$

where

$$\sqrt{g}\, j^\varrho = B_{l,\varphi} - B_{\varphi,l} ,$$

$$\sqrt{g}\, j^\varphi = B_{\varrho,l} - B_{l,\varrho} ,$$

$$\sqrt{g}\, j^l = B_{\varphi,\varrho} - B_{\varrho,\varphi} ,$$

are supplemented by the condition for magnetic surfaces,

$$B^\varrho V_{,\varrho} + B^\varphi V_{,\varphi} + B^l V_{,l} = 0 . \tag{7 d}$$

It is easy to see that the third component of the momentum balance is satisfied if Eqs. (7 b – d) are solved and if $B^l \neq 0$. The equilibrium calculation is performed in Appendix I. The first set of results is

$$\begin{aligned} c_1 &= 2\varkappa c_0\cos\varphi , \\ a_1 &= a_{10} + a_{1c}\cos 2u + a_{1s}\sin 2u , \\ b_0 &= b_{00} + a_{1s}\cos 2u - a_{1c}\sin 2u , \\ V_2 &= V_{20} + V_{2c}\cos 2u , \\ c_2 &= 2\tau b_0 + 3c_0\varkappa^2\cos^2\varphi - c_0\tau^2 - b_{00}b_0/c_0 \\ &\quad -\tfrac{1}{4}(b_{0,\varphi l} + c_0'') - \dot{p}\, V_2/c_0 \end{aligned} \tag{8}$$

with

$$\begin{aligned} a_{10} &= -\tfrac{1}{2}c_0' , \quad b_{00} = \tfrac{1}{2}j + \tau c_0 , \\ a_{1s} &= (b_{00} + c_0\alpha')\frac{e^2-1}{e^2+1} , \\ a_{1c} &= -c_0 e'/2e , \\ V_{20} &= \frac{\pi}{2} I c_0(e + 1/e) , \\ V_{2c} &= \frac{\pi}{2} I c_0(e - 1/e) , \quad I = \oint \mathrm{d}l/c_0 , \end{aligned} \tag{9}$$

where the prime indicates the derivative with respect to the arc length l and j is the current density on the magnetic axis. Instead of b_{00} we shall also use the quantity

$$K_0' = (j/c_0 + 2\tau + 2\alpha')/(e + 1/e)\,, \tag{10}$$

which is related to the rotational transform ι by

$$2\pi\iota = K_0(L) - K_0(0) - \alpha(L) + \alpha(0) - \oint \tau\, dl \tag{11}$$

[where $\alpha(L) - \alpha(0) = n\pi$ and $\oint \tau\, dl = 2\pi m$, where m is the number of full turns of the normal over the length of the magnetic axis], and in terms of which

$$b_{00} = c_0[\tfrac{1}{2} K_0'(e + 1/e) - \alpha']\,,$$
$$a_{1s} = \tfrac{1}{2} c_0 K_0'(e - 1/e)\,.$$

Those parts of the second set of results which are necessary for the evaluation of the magnetic well may be expressed in terms of b_1 and relations connecting the third-order coefficients, δ, Δ, S, and s. One obtains

$$b_1 = b_{11} + b_{13}\,,$$
$$b_{11} = b_{11s}(l) \sin u + b_{11c}(l) \cos u\,,$$
$$b_{13} = b_{13s} \sin 3u + b_{13c} \cos 3u\,,$$

$$b_{11s} = \overline{b}_{11s} + \tfrac{1}{16}\{2\varkappa \sin\alpha[12\tau c_0 + c_0(-5\alpha' + 2K_0' e + 3K_0'/e)] + \cos\alpha(c_0 e' \varkappa/e + 5 c_0' \varkappa + 2 c_0 \varkappa')\}\,, \tag{12 a}$$

$$b_{11c} = \overline{b}_{11c} + \tfrac{1}{16}\{2\varkappa \cos\alpha[12\tau c_0 + c_0(-5\alpha' + 3K_0' e + 2K_0'/e)] + \sin\alpha(c_0 e' \varkappa/e - 5 c_0' \varkappa - 2 c_0 \varkappa')\}\,, \tag{12 b}$$

$$\overline{b}_{11s} = \tfrac{3}{2}\, \dot{p}\, \pi I c_0^{3/2} e^{-1/2} (b_r \cos\alpha + b_i \sin\alpha)\,, \quad \overline{b}_{11c} = \tfrac{3}{2}\, \dot{p}\, \pi I c_0^{3/2} e^{+1/2} (b_i \cos\alpha - b_r \sin\alpha)\,, \tag{13}$$

where the complex quantity $b = b_r + i\, b_i$ solves

$$b' + i(K_0' - \alpha') b = -\exp(i\alpha)\, c_0^{-3/2} \varkappa\, (e^{-1/2} \cos\alpha - i\, e^{1/2} \sin\alpha) \tag{14}$$

with the boundary conditions $b(L) = b(0)$. Introducing the notations

$$\overline{S}_c = S\cos\alpha - s\sin\alpha\,, \quad \overline{S}_s = S\sin\alpha + s\cos\alpha \tag{15}$$

one obtains

$$\left(e + \frac{3}{e}\right)\left[-\frac{1}{2}\left(\frac{c_0'}{c_0} + \frac{e'}{e}\right)\overline{S}_c + \overline{S}_c' + \frac{1}{e} K_0' \overline{S}_s\right] + \frac{1}{2}\left(\frac{c_0'}{c_0} - \frac{e'}{e}\right)\delta - \delta' - K_0' \Delta = R_1\,,$$
$$\left(3e + \frac{1}{e}\right)\left[-\left(\frac{1}{2}\frac{c_0'}{c_0} + \frac{3}{2}\frac{e'}{e}\right)\overline{S}_s + \overline{S}_s' - e K_0' \overline{S}_c\right] - e^2\left[-\frac{1}{2}\left(\frac{c_0'}{c_0} + \frac{e'}{e}\right)\Delta + \Delta'\right] + e K_0' \delta = R_2\,, \tag{16}$$

with

$$R_1 = \frac{L\varkappa}{8\pi}\left\{2\sin\alpha\left[-K_0' + \alpha'\left(3e - \frac{2}{e}\right) + 4\tau e\right]\right.$$
$$\left. + \cos\alpha\left[\frac{c_0'}{c_0}\left(e - \frac{2}{e}\right) + 2\frac{\varkappa'}{\varkappa}\left(e + \frac{2}{e}\right) - \frac{e'}{e}\left(3e + \frac{2}{e}\right)\right] + \frac{16}{3}\frac{e}{\varkappa c_0}\overline{b}_{11s}\right\},$$
$$R_2 = e^2 \frac{L\varkappa}{8\pi}\left\{2\cos\alpha\left[K_0' + \alpha'\left(2e - \frac{3}{e}\right) - 4\tau/e\right]\right. \tag{17}$$
$$\left. + \sin\alpha\left[\frac{c_0'}{c_0}\left(\frac{1}{e} - 2e\right) + 2\frac{\varkappa'}{\varkappa}\left(2e + \frac{1}{e}\right) + \frac{e'}{e}\left(2e + \frac{3}{e}\right)\right] - \frac{16}{3}\frac{1}{e\varkappa c_0}\overline{b}_{11c}\right\},$$

Equations (10), (11), (14), and (16) [or Eq. (A 31) of Appendix I] may be used to discuss the consequences of the rotational transform being integer (see e. g. [1, 6]).

Finally, we use the result for c_2, Eqs. (8), and reduce the expression for the magnetic well, Eq. (6), to the following form:

$$\ddot{\Phi}/\dot{\Phi} = \frac{1}{2\pi(\oint dl/c_0)^2} \cdot \oint \frac{dl}{c_0^2}\left\{\frac{\varkappa^2}{2}\left[e + \frac{1}{e} - \left(e - \frac{1}{e}\right)\cos 2\alpha\right] - \left(\frac{j}{c_0}\right)^2 \frac{1}{e + 1/e}\right.$$
$$\left. - \frac{(e - 1/e)^2}{e + 1/e}(\tau + \alpha')^2 - \frac{3}{4}\left(\frac{c_0'}{c_0}\right)^2\left(e + \frac{1}{e}\right) - \frac{1}{4}\left(\frac{e'}{e}\right)^2\left(e + \frac{1}{e}\right) + \frac{c_0'}{c_0}\frac{e'}{e}\left(e - \frac{1}{e}\right)\right\}$$
$$- \frac{1}{\oint dl/c_0}\, \dot{p} \oint \frac{dl}{c_0^3} - \frac{2}{L(\oint dl/c_0)^2} \oint \frac{\varkappa}{c_0^2}\left[\frac{4}{e} S - (\Delta \sin\alpha + \delta\cos\alpha)\right] dl\,. \tag{18}$$

IV. Stability Criteria

We employ the following stability criteria. The sufficient criterion[7] for a configuration with non-vanishing current density on the magnetic axis reads (for $\dot p<0$)

$$\ddot{\Phi}/\dot{\Phi}+\dot p\,|\nabla\zeta|^2>0\,. \qquad (19)$$

This criterion applies to the case of a perfectly conducting wall at the plasma boundary and has been used before to discuss the stability of axially symmetric equilibria[8, 9]. The sufficient criterion for a configuration with vanishing current density on the magnetic axis is

$$\ddot{\Phi}/\dot{\Phi}+\dot p\oint|\nabla\zeta|^2\,\mathrm{d}\zeta>0\,, \qquad (20)$$

which is less restrictive than inequality (19) and applies without a perfectly conducting wall at the plasma boundary. This criterion is an improvement of earlier sufficient criteria[10] and is derived in Appendix III. The necessary criterion[1] is used in the form[8]

$$\ddot{\Phi}/\dot{\Phi}+\dot p\int g_{\Theta\Theta}\,|\nabla V|^{-2}\,\mathrm{d}\Theta\,\mathrm{d}\zeta>0\,. \qquad (21)$$

In the above formulae V, Θ, ζ are Hamada coordinates. In Appendix II we introduce these coordinates and reduce the criteria. The results may be described in terms of the quantities ξ_c, ξ_s, η_c, η_s, l_c, l_s, which are defined as follows: ξ_c, ξ_s, η_c, η_s are given by

$$\xi_c=(\pi\,l_{0,\zeta})^{-1/2}\,(e^{1/2}\sin\alpha\sin K+e^{-1/2}\cos\alpha\cos K)\,,$$

$$\xi_s=(\pi\,l_{0,\zeta})^{-1/2}\,(e^{1/2}\sin\alpha\cos K-e^{-1/2}\cos\alpha\sin K)\,,$$

$$\eta_c=-(\pi\,l_{0,\zeta})^{-1/2}\,(e^{-1/2}\sin\alpha\cos K - e^{1/2}\cos\alpha\sin K)\,,$$

$$\eta_s=(\pi\,l_{0,\zeta})^{-1/2}\,(e^{-1/2}\sin\alpha\sin K+e^{1/2}\cos\alpha\cos K)\,,$$

where $l_0(\zeta)$ is the arc length as a function of ζ in lowest order in the distance from the magnetic axis, $l_{0,\zeta}=c_0 I$, and $K'(l)=K_0'(l)-2\pi\iota/c_0 I$. The quantities $l_c(\zeta)$ and $l_s(\zeta)$ obey the equations

$$l_{c,\zeta}-l_{0,\zeta\zeta}\,l_c/l_{0,\zeta}+2\pi\iota\,l_s=2\varkappa\,l_{0,\zeta}\,\xi_c\,,$$

$$l_{s,\zeta}-l_{0,\zeta\zeta}\,l_s/l_{0,\zeta}-2\pi\iota\,l_c=2\varkappa\,l_{0,\zeta}\,\xi_s\,.$$

The results are

$$|\nabla\zeta|^2=\frac{1}{l_{0,\zeta}^2}+\pi^2\left[(\eta_s l_c-\eta_c l_s)^2+(\xi_s l_c-\xi_c l_s)^2\right]\,, \qquad (22)$$

$$\int g_{\Theta\Theta}\,|\nabla V|^{-2}\,\mathrm{d}\Theta=\frac{1}{l_{0,\zeta}^2}\left\{1+\frac{\pi\,l_{0,\zeta}}{2}\left[l_c^2+l_s^2-\frac{(l_s^2-l_c^2)(\eta_s^2-\eta_c^2+\xi_s^2-\xi_c^2)+4\,l_c\,l_s\,(\eta_c\eta_s+\xi_c\xi_s)}{\dfrac{2}{\pi\,l_{0,\zeta}}+\eta_c^2+\eta_s^2+\xi_c^2+\xi_s^2}\right]\right\}. \qquad (23)$$

Note that the criteria (20) and (21) are such that the second term in the expression for the magnetic well, Eq. (18), cancels with the first term in Equations (22), (23). Also note that the necessary criterion and the sufficient criterion (20) are identical if $\varkappa\equiv 0$, i.e. for a straight magnetic axis. We therefore obtain the result that a straigth equilibrium with $j=\tau=\alpha'=0$ can be stable for $e>2+\sqrt{3}$ (with respect to the necessary criterion this was found before; see[1, 11]). From Eqs. (16), (17) it is also clear that, for a straigth magnetic axis, $S=s=\delta=\Delta=0$ is a solution of the third-order equations. We conclude that there is no limitation for the plasma-β of the stable configuration constructed above within the framework of the third-order expansion around the magnetic axis since there are no stagnation points in the flux surfaces in this case.

V. Stagnation Point Discussion and Beta Estimate

It is easy to see that the explicit dependence on α in Eqs. (14), (16), (17) can be removed by introducing

$$b=[\bar b_{rc}\cos\alpha+\bar b_{rs}\sin\alpha+i(\bar b_{ic}\cos\alpha+\bar b_{is}\sin\alpha)]\cdot\exp(i\alpha)\,,$$

$$\delta=\delta_c\cos\alpha+\delta_s\sin\alpha\,,$$

$$\Delta=\Delta_c\cos\alpha+\Delta_s\sin\alpha\,, \qquad (24)$$

$$\bar S_c=S_{cc}\cos\alpha+S_{cs}\sin\alpha\,,$$

$$\bar S_s=S_{sc}\cos\alpha+S_{ss}\sin\alpha\,.$$

Considering the expressions for the volume Eqs. (A 26), (A 28) at the values $\alpha=0,\pi/2$ (where the elliptical cross-section aligns its half-axes with the normal and binormal of the magnetic axis), one finds that $\delta_s=\Delta_c=S_{cs}=S_{sc}=0$ is equivalent to the cross-section being symmetric with respect to the osculating plane. There are many physically interesting cases (e, g. axially and helically symmetric, $l=2$ stellarator, and $\iota=0$ M & S equilibria) where solutions with this property may be found. Since there is no indication that this property is detrimental to equilibrium and stability properties, we restrict our discussion to cases where this symmetry holds:

$$\delta=\delta_c\cos\alpha\,,\qquad \Delta=\Delta_s\sin\alpha\,,$$

$$\bar S_c=S_c\cos\alpha\,,\qquad \bar S_s=S_s\sin\alpha\,. \qquad (25)$$

One then obtains

$$V = \frac{L^2}{2\pi} I c_0 \left[\frac{1}{2} e x^2 + \frac{1}{2e} y^2 + x^3 e S_c \cos\alpha + x^2 y \left(\frac{S_s}{e} - \Delta_s \right) \sin\alpha + x y^2 \left(\frac{S_c}{e} - \delta_c \right) \cos\alpha + y^3 \frac{S_s}{e^3} \sin\alpha \right], \tag{26}$$

where dimensionless variables x and y have been introduced:

$$x = \frac{2\pi}{L} \varrho \cos u, \quad y = \frac{2\pi}{L} \varrho \sin u.$$

We now discuss the large aspect ratio properties of Eqs. (13), (16), (17), (26). From Eq. (26) we obtain for the volume V_s of a magnetic surface on which a stagnation point lies

$$V_s \sim L^3/A^2, \quad \Sigma = (S_c, S_s, \delta_c, \Delta_s) \sim A,$$

where A measures the aspect ratio. Two different cases have to be considered.

Case I: The number n of field periods is fixed, $n \sim O(1)$. Then $\Sigma' \sim A/L$, provided that $|p L^3/c_0^2| \lesssim O(A)$. Defining the plasma β by

$$\beta = \int_0^{V_s} p \, dV \Big/ \int_0^{V_s} \tfrac{1}{2} B^2 \, d^3\tau \approx - \frac{p V_s}{\dot{\Phi}_0 \oint c_0 \, dl}, \tag{27}$$

we therefore obtain

$$\beta \sim \left| \frac{p L^3}{c_0^2} \right| \Big/ A^2, \tag{28}$$

i.e. $\beta \sim 1/A$. This situation is typical of tokamak and stellarator equilibria with finite rotational transform $[\iota \sim 0(1)]$.

Case II: The number n of field periods is $O(A)$. Because of Eq. (14) two subcases depending on the magnitude of the curvature are obtained.

Case IIa: $\varkappa \sim n/L$. Then $\Sigma' \sim nA/L$ provided that $|p L^3/c_0^2| \lesssim A n$. So, $\beta \sim 1$ in this case, which is typical of, for example, M & S equilibria.

Case IIb: $\varkappa \sim 1/L$. Then $\Sigma' \sim nA/L$, provided that $|p L^3/c_0^2| \lesssim A n^2$. Thus, $\beta \sim A$ in this case, which is typical of $l = 2$ stellarator equilibria with a large number of field periods $[n \sim O(A)]$.

We briefly mention the additional restriction imposed by taking into consideration one of the stability criteria (19) – (21). We restrict our discussion to cases without longitudinal current on the magnetic axis or $c_0 = \text{const}$ if there is a longitudinal current on the magnetic axis, because, for these cases, the diamagnetic deepening of the magnetic well [the second last term in Eq. (18)] cancels with the first term in Equations (22), (23). One then finds that the stability criteria (19) – (21) obey the same scaling with aspect ratio. Using Eq. (18), one easily verifies that for each of the cases I, IIa, IIb there are stable equilibria. For Case IIb this is of course the toroidally closed equilibrium discussed in Section IV.

We conclude this section with a discussion of the cross-section of the magnetic surfaces given by Eq. (2) at $\alpha = 0, \pi/2$. This will provide bounds for the aspect ratio and the volume V_s. For this particular purpose it is convenient to introduce the following transformation (the so-called "rounded coordinate system" [3]):

$$x = \tilde{x}/\sqrt{e}, \quad S_c = \tilde{S}_c \sqrt{e}, \quad \delta_c = \tilde{\delta}_c/\sqrt{e},$$
$$y = \tilde{y}\sqrt{e}, \quad S_s = \tilde{S}_s \sqrt{e^3}, \quad \Delta_s = \tilde{\Delta}_s \sqrt{e}.$$

Then

$$V(\alpha = 0) = \frac{L^2}{2\pi} I c_0 [\tfrac{1}{2}(\tilde{x}^2 + \tilde{y}^2) + \tilde{x}^3 \tilde{S}_c + \tilde{x}\tilde{y}^2 (\tilde{S}_c - \tilde{\delta}_c)],$$

$$V(\alpha = \pi/2) = \frac{L^2}{2\pi} I c_0 [\tfrac{1}{2}(\tilde{x}^2 + \tilde{y}^2) + \tilde{y}^3 \tilde{S}_s + \tilde{x}^2 \tilde{y} (\tilde{S}_s - \tilde{\Delta}_s)],$$

so that it is sufficient to discuss the one parameter family of normalized flux functions

$$T = \tfrac{1}{2}(v^2 + w^2) + v^3 + v w^2 D, \tag{29}$$

where $v = \tilde{x}\tilde{S}_c$, $w = \tilde{y}\tilde{S}_c$, $D = 1 - \tilde{\delta}_c/\tilde{S}_c$ for $\alpha = 0$ and $v = \tilde{y}\tilde{S}_s$, $w = \tilde{x}\tilde{S}_s$, $D = 1 - \tilde{\Delta}_s/\tilde{S}_s$ for $\alpha = \pi/2$. The following intervals have to be distinguished:

$$\begin{aligned} &D > 3/2, && T_s = (D-1)/(8D^3) < 1/54, \\ &3/2 \geqq D \geqq -3, && T_s = 1/54, \\ &-3 > D, && T_s = (D-1)/(8D^3) < 1/54. \end{aligned} \tag{30}$$

There is always a stagnation point at $v = -1/3$, $w = 0$ and a straigth field line of the transverse field at $v = -1/(2D)$. (Representative cases are plotted in Figure 1.) Because of Eqs. (30) the following bounds are obtained for the aspect ratio and the volume V_s:

$$A \geqq 3 \max(|S_c|, |S_s|/e^2), \tag{31}$$

$$V_s \leqq \frac{1}{54} \frac{L^2}{2\pi} I c_0 \min(e/S_c^2, e^3/S_s^2).$$

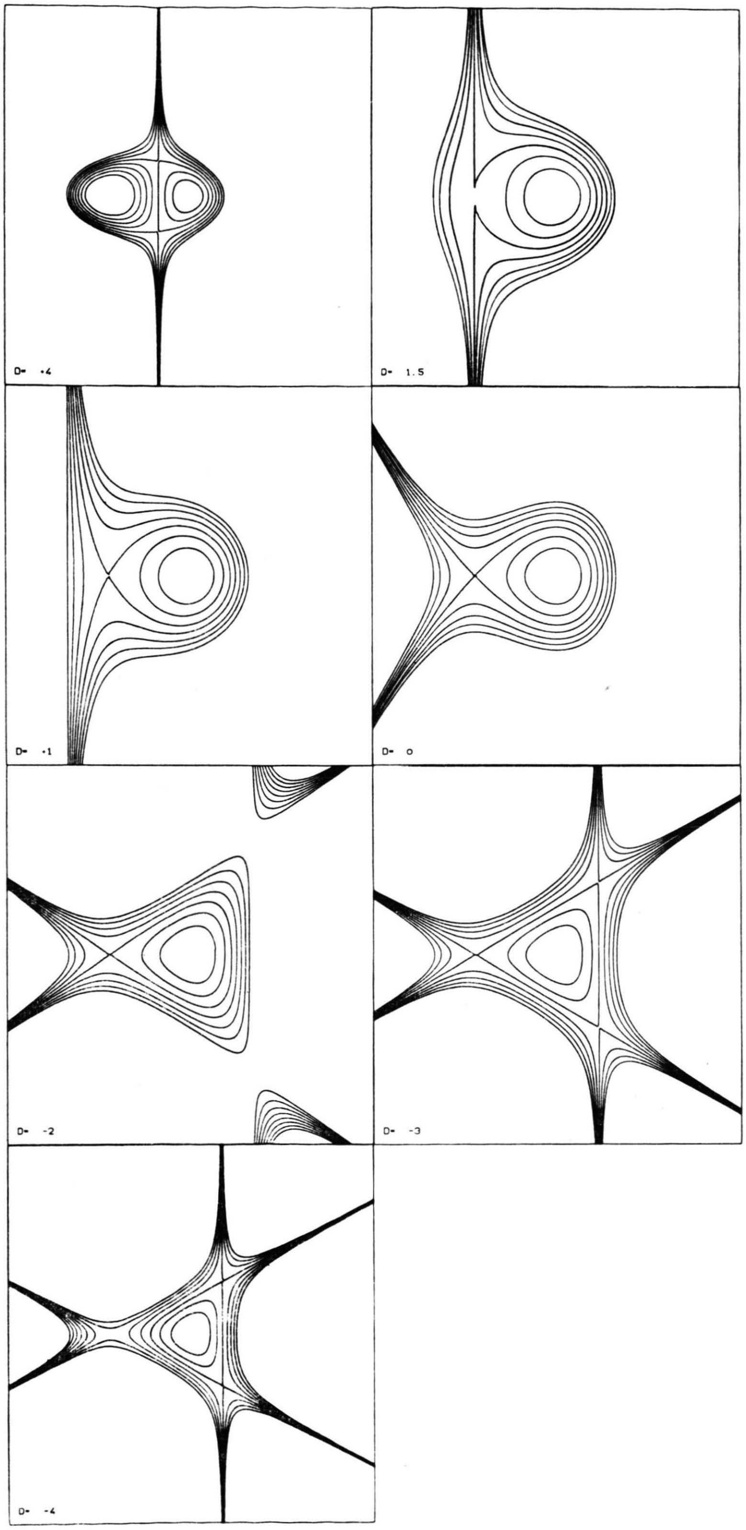

Fig. 1. Contour lines of the function $\frac{1}{2}(v^2+w^2)+v^3+v\,w^2\,D$ for various values of D. The v-axis points to the right, the w-axis upward. There is always a straight contour line at $v=-1/(2\,D)$ and a stagnation point at $v=-\frac{1}{3}$, $w=0$. Special cases are $D=\frac{3}{2}$, 0, -3, where a degenerate stagnation point, no straight contour line, and three stagnation points on one contour line occur respectively.

We stress that taking into account the limits imposed by Ineqs. (31) is by no means sufficient to obtain valid β estimates according to Equation (27). The third-order equilibrium equations Eqs. (16), (17) impose additional restrictions since they connect $\dot{p}$, S_c, S_s, δ_c and Δ_s. In order to obtain a correct β estimate, the following procedure has to be observed. Either S_c, S_s or δ_c, Δ_s and $\dot{p}$ may be prescribed; then δ_c, Δ_s or (S_c, S_s) are calculated according to Equations (16), (17). The stagnation points of the third-order flux surfaces Eq. (26) are calculated for each value of α, which leads to a maximum admissible volume $V_s(\alpha)$ [an upper estimate for this quantity is provided by Eq. (31)]; the minimum value of the volumes $V_s(\alpha)$ has to be inserted in Eq. (27); to eliminate the arbitrary choices for S_c, S_s and $\dot{p}$, the β-value has to be optimized with respect to these quantities. Only this optimized β-value may be considered as a quantity which characterizes a basic property of the configuration which is studied. If one wants to obtain a β estimate taking into account a stability criterion, an additional inequality resulting from one of the Eqs. (19) – (21) for the quantities S_c, S_s, and $\dot{p}$ has to be satisfied.

VI. Conclusion

In this paper we have established a consistent procedure to obtain an estimate for equilibrium and stability β-values in three-dimensional toroidal equilibria. In particular the evaluation of a stability criterion is not sufficient to obtain a β-estimate. Rather, the stability criteria have to be used as side conditions for the equilibrium β-estimate which in turn is obtained from a discussion of stagnation points in the third order flux surfaces.

The β-values obtainable by this procedure are estimates mainly because the stagnation points in the magnetic surfaces may shift if the equilibrium theory is refined (for example, by taking into account fourth order terms or by considering exact equilibria). Our experience with axially symmetric equilibria [9] has been that, if the third order theory predicts a separatrix near to the magnetic axis, the results are only weakly changed by considering exact equilibria.

As an application and an example of a three-dimensional configuration we have already treated the $l=2$ stellarator with circular magnetic axis, without longitudinal current on the magnetic axis, and with uniformly rotating elliptical cross-section. Since rather lengthy calculations and optimizations have to be done to obtain unequivocal results, we reserve the details to a subsequent paper. The important result is that the bounds in the equilibrium and stability β-values are lower than has commonly been assumed. Irrespective of the parameters describing this type of stellarator, there are upper limits of 0.66% and 0.22% for the necessary and the sufficient stability criterion respectively. These results can be extended in several ways. Including a longitudinal current will close the gap between tokamak results (see, for example [8]) and the results for the $l=2$ stellarator. Admitting a helix-like toroidally closed magnetic axis will probably augment the β-values obtainable (see [2]). Also, a systematic study of helically symmetric equilibria (see for example [12]) has still to be made. Toroidal $\iota=0$ equilibria (see, for example [13]) also deserve a further study which is readily feasible in view of the compact formulae (16), (17).

Probably the most interesting question within the framework of ideal MHD equilibrium and stability will be whether there exist classes of stable equilibria with β-values sufficiently larger than achievable in axially symmetric devices in order to make three-dimensional toroidal equilibria attractive despite their complexity.

Appendix I

Here we account for the equilibrium calculation leading to the results of Section III. Using the metric given by Eq. (1) and the notation for the contravariant components of $\mathbf{B}$ given by Eq. (2), one obtains

$$
\begin{aligned}
B_\varrho &= g_{\varrho\varrho} B^\varrho = a_1 \varrho + a_2 \varrho^2 + O(\varrho^3)\,, \quad B_\varphi = g_{\varphi\varphi} B^\varphi + g_{\varphi l} B^l = (b_0 - \tau c_0)\varrho^2 + (b_1 - \tau c_1)\varrho^3 + O(\varrho^4)\,,\\
B_l &= g_{\varphi l} B^\varphi + g_{ll} B^l = c_0 + \varrho(c_1 - 2 c_0 \varkappa \cos\varphi)\\
&\quad + \varrho^2[-\tau b_0 + c_2 - 2 c_1 \varkappa\cos\varphi + c_0(\varkappa^2\cos^2\varphi + \tau^2)]\\
&\quad + \varrho^3[-\tau b_1 + c_3 - 2 c_2 \varkappa\cos\varphi + c_1(\varkappa^2\cos^2\varphi + \tau^2)] + O(\varrho^4)\,,
\end{aligned}
\tag{A 1}
$$

and
$$\sqrt{g}\, j^{\varrho} = B_{l,\varphi} - B_{\varphi,l} = \varrho\,(c_{1,\varphi} + 2\,c_0\,\varkappa \sin\varphi) + \varrho^2\,[-\tau\, b_{0,\varphi} + c_{2,\varphi} - 2\,c_{1,\varphi}\,\varkappa\cos\varphi + 2\,c_1\,\varkappa\sin\varphi - 2\sin\varphi\cos\varphi\, c_0\,\varkappa^2 - b_{0,l} + (\tau\, c_0)'] + \varrho^3\,[-\tau\, b_{1,\varphi} + c_{3,\varphi} - 2\,c_{2,\varphi}\,\varkappa\cos\varphi + 2\,c_2\,\varkappa\sin\varphi - 2\,c_1\,\varkappa^2\sin\varphi\cos\varphi + c_{1,\varphi}\,(\varkappa^2\cos^2\varphi + \tau^2) - b_{1,l} + (\tau\, c_1)_{,l}] + O(\varrho^4)\,,$$

$$\sqrt{g}\, j^{\varphi} = B_{\varrho,l} - B_{l,\varrho} = -\,(c_1 - 2\,c_0\,\varkappa\cos\varphi) + \varrho\,\{a_{1,l} - 2\,[-\tau\, b_0 + c_2 - 2\,c_1\,\varkappa\cos\varphi + c_0\,(\varkappa^2\cos^2\varphi + \tau^2)]\} + \varrho^2\,\{a_{2,l} - 3\,[-\tau\, b_1 + c_3 - 2\,c_2\,\varkappa\cos\varphi + c_1\,(\varkappa^2\cos^2\varphi + \tau^2)]\} + O(\varrho^3)\,, \qquad \text{(A 2)}$$

$$\sqrt{g}\, j^{l} = B_{\varphi,\varrho} - B_{\varrho,\varphi} = \varrho\,(2\,b_0 - 2\,\tau\, c_0 - a_{1,\varphi}) + \varrho^2\,(3\,b_1 - 3\,\tau\, c_1 - a_{2,\varphi}) + O(\varrho^3)\,.$$

We conclude that
$$c_1 = 2\,\varkappa\, c_0 \cos\varphi \qquad \text{(A 3)}$$
from the analyticity of the current density and introduce
$$j = 2\,b_0 - 2\,\tau\, c_0 - a_{1,\varphi}\,, \qquad \text{(A 4)}$$
where j is the current density on the magnetic axis. We now obtain the developments with respect to ϱ of Equations (7 a – d):

$O(\varrho)$ of Eq. (7a):
$$2\,a_1 + b_{0,\varphi} + c_0' = 0\,, \qquad \text{(A 5)}$$

$O(\varrho^2)$ of Eq. (7a):
$$3\,a_2 + b_{1,\varphi} - 3\,a_1\,\varkappa\cos\varphi + b_0\,\varkappa\sin\varphi - \varkappa\cos\varphi\, b_{0,\varphi} - c_0\,\varkappa'\cos\varphi + c_{1,l} - c_0'\,\varkappa\cos\varphi = 0\,, \qquad \text{(A 6)}$$

$O(\varrho)$ of Eq. (7b):
$$2\,\dot p\, V_2 = c_0\, a_{1,l} - 2\,c_0\,(-\tau\, b_0 + c_2 - 3\,c_0\,\varkappa^2\cos^2\varphi + c_0\,\tau^2) - b_0\,(2\,b_0 - a_{1,\varphi} - 2\,\tau\, c_0)\,, \qquad \text{(A 7)}$$

$O(\varrho^2)$ of Eq. (7b):
$$3\,\dot p\, V_3 = 2\,c_0\,\varkappa\cos\varphi\,(a_{1,l} + 5\,\tau\, b_0 + c_2 + 3\,c_0\,\varkappa^2\cos^2\varphi - 5\,\tau^2\, c_0) - b_1\,(5\,b_0 - 5\,\tau\, c_0 - a_{1,\varphi}) + c_0\,(a_{2,l} - 3\,c_3) + b_0\, a_{2,\varphi}\,, \qquad \text{(A 8)}$$

$O(\varrho^2)$ of Eq. (7c):
$$\dot p\, V_{2,\varphi} = a_1\,(2\,b_0 - 2\,\tau\, c_0 - a_{1,\varphi}) + c_0\,(\tau\, b_{0,\varphi} - c_{2,\varphi} - 6\,\varkappa^2\, c_0 \sin\varphi\cos\varphi + b_{0,l} - c_0\,\tau' - c_0'\,\tau)\,, \qquad \text{(A 9)}$$

$O(\varrho^3)$ of Eq. (7c):
$$\dot p\, V_{3,\varphi} = a_2\,(2\,b_0 - 2\,\tau\, c_0 - a_{1,\varphi}) + a_1\,(3\,b_1 - 6\,\varkappa\,\tau\, c_0\cos\varphi - a_{2,\varphi}) + 2\,c_0\,\varkappa\cos\varphi\,(\tau\, b_{0,\varphi} + b_{0,l}) - c_0\,(-\tau\, b_{1,\varphi} + c_{3,\varphi} + 2\,c_2\,\varkappa\sin\varphi + 6\sin\varphi\cos^2\varphi\, c_0\,\varkappa^3 - 2\,c_0\,\varkappa\,\tau^2\sin\varphi - b_{1,l}) - 2\,c_0\cos\varphi\,(2\,\tau'\, c_0\,\varkappa + 2\,\tau\, c_0'\,\varkappa + \tau\, c_0\,\varkappa')\,, \qquad \text{(A 10)}$$

$O(\varrho^2)$ of Ep. (7d):
$$2\,a_1\, V_2 + b_0\, V_{2,\varphi} + c_0\, V_{2,l} = 0\,; \qquad \text{(A 11)}$$

$O(\varrho^3)$ of Eq. (7d):
$$3\,a_1\, V_3 + 2\,a_2\, V_2 + b_0\, V_{3,\varphi} + b_1\, V_{2,\varphi} + c_0\, V_{3,l} + c_1\, V_{2,l} = 0\,. \qquad \text{(A 12)}$$

Considering Eqs. (A 4), (5), (11) and introducing the flux surface geometry up to second order by [according to Eqs. (4), (5)]
$$V_2 = V_{20} + V_{2c}\cos 2\,u$$
with
$$V_{20} = \tfrac{1}{2}\,\pi\, I\, c_0\,(e + 1/e)\,, \quad V_{2c} = \tfrac{1}{2}\,\pi\, I\, c_0\,(e - 1/e)\,, \quad I = \oint \mathrm{d}l/c_0 \qquad \text{(A 13)}$$
one finds that Eqs. A 4), (5), (11) are solved by
$$b_0 = b_{00} + b_{0c}\cos 2\,u + b_{0s}\sin 2\,u\,, \quad a_1 = a_{10} - b_{0s}\cos 2\,u + b_{0c}\sin 2\,u \qquad \text{(A 14)}$$
where
$$b_{00} = \tfrac{1}{2}\, j + \tau\, c_0\,, \quad a_{10} = -\tfrac{1}{2}\, c_0'\,, \quad b_{0c} = (b_{00} + c_0\,\alpha')\,\frac{e^2 - 1}{e^2 + 1}\,, \quad b_{0s} = \frac{c_0\, e'}{2\,e}\,, \qquad \text{(A 15)}$$
or, if we introduce the quantity K_0',
$$K_0' = (j/c_0 + 2\,\tau + 2\,\alpha')/(e + 1/e)\,, \quad b_{00} = c_0[\tfrac{1}{2}\, K_0'\,(e + 1/e) - \alpha']\,, \quad b_{0c} = \tfrac{1}{2}\, c_0\, K_0'\,(e - 1/e)\,. \qquad \text{(A 16)}$$
We now obtain c_2 from Eq. (A 7):
$$c_2 = 2\,\tau\, b_0 + 3\,c_0\,\varkappa^2\cos^2\varphi - c_0\,\tau^2 - b_{00}\, b_0/c_0 - \tfrac{1}{4}\,(b_{0,\varphi l} + c_0'') - \dot p\, V_2/c_0\,, \qquad \text{(A 17)}$$

Equation (A 9) then reduces to
$$j/c_0 = \text{const.} \qquad \text{(A 18)}$$

The remaining Eqs. (A 6), (8), (10), (12) are treated as follows. Using the previous results, we obtain a_2 in terms of b_1, from Eq. (A 6):

$$a_2 = I_1 - \frac{1}{3}\, b_{1,\varphi}\,, \quad I_1 = I_c \cos u + I_s \sin u\,,$$
$$I_c = -\frac{1}{3}\left[-c_0\,\varkappa \sin a (e\,K_0' - a') + \frac{1}{2}\left(c_0\,\frac{e'}{e}\,\varkappa + 5\,c_0'\,\varkappa + 2\,c_0\,\varkappa' \right) \cos a \right], \qquad \text{(A 19)}$$
$$I_s = -\frac{1}{3}\left[c_0\,\varkappa \cos a \left(\frac{K_0'}{e} - a' \right) + \frac{1}{2}\left(-c_0\,\frac{e'}{e}\,\varkappa + 5\,c_0'\,\varkappa + 2\,c_0\,\varkappa' \right) \sin a \right].$$

The quantities c_3 and V_3 are eliminated from Equations (A 8), (10). The resulting equation contains only a_2 and b_1, so that an equation for b_1 can be obtained by inserting a_2 from Equation (A 19). The equation for b_1 is

$$-c_0(3\,b_{1,l} + \tfrac{1}{3}\,b_{1,\varphi\varphi l}) + \tfrac{1}{2}(b_{0,\varphi} + 3\,c_0')\,(3\,b_1 + \tfrac{1}{3}\,b_{1,\varphi\varphi}) - b_0\,(3\,b_{1,\varphi} + \tfrac{1}{3}\,b_{1,\varphi\varphi\varphi}) = R\,,$$
$$\begin{aligned} R = \sin\varphi\,[&\tfrac{5}{3}\,\varkappa\,b_0^2 + \tfrac{1}{2}\,\varkappa\,b_0\,b_{0,\varphi\varphi} + 4\,\varkappa\,\tau\,c_0\,b_0 + 4\,\varkappa\,p\,V_2 + \tfrac{1}{3}\,\varkappa'\,c_0\,b_{0,\varphi} + \tfrac{1}{6}\,\varkappa\,c_0\,b_{0,\varphi l} \\ &+ \tfrac{1}{6}\,c_0'\,\varkappa\,b_{0,\varphi} - \tfrac{1}{12}\,\varkappa\,b_{0,\varphi}{}^2 - \tfrac{2}{3}\,c_0\,c_0'\,\varkappa' + \tfrac{5}{4}\,c_0'^2\,\varkappa - \tfrac{5}{6}\,c_0\,c_0''\,\varkappa - \tfrac{1}{3}\,c_0^2\,\varkappa''] \\ + \cos\varphi\,[&\tfrac{5}{3}\,\varkappa\,c_0\,b_{0,\varphi\varphi l} + 2\,c_0\,\varkappa\,\tau\,b_{0,\varphi} + 2\,\tau\,c_0\,c_0'\,\varkappa + \tfrac{19}{3}\,c_0\,\varkappa\,b_{0,l} - 12\,\tau'\,c_0^2\,\varkappa - 4\,c_0^2\,\tau\,\varkappa' \\ &+ 2\,\varkappa\,p\,V_{2,\varphi} + \tfrac{2}{3}\,\varkappa\,b_0\,b_{0,\varphi} + \tfrac{1}{6}\,\varkappa\,b_{0,\varphi}\,b_{0,\varphi\varphi} - 2\,c_0\,\varkappa'\,b_0 - \tfrac{1}{3}\,c_0\,\varkappa'\,b_{0,\varphi\varphi} - \tfrac{19}{3}\,\varkappa\,c_0'\,b_0 - \tfrac{3}{2}\,c_0'\,\varkappa\,b_{0,\varphi\varphi}]\,. \end{aligned} \qquad \text{(A 20)}$$

The decomposition
$$b_1 = b_{11} + b_{13}\,, \quad b_{11} = b_{11s}(l)\sin u + b_{11c}(l)\cos u\,,$$
$$b_{13} = b_{13s}(l)\sin 3\,u + b_{13c}(l)\cos 3\,u \qquad \text{(A 21)}$$
shows that Eq. (A 20) is merely an equation for b_{11}:
$$-2\,c_0\,b_{11,l}|_\varphi + (b_{0,\varphi} + 3\,c_0')\,b_{11} - 2\,b_0\,b_{11,\varphi} = \tfrac{3}{4}\,R\,.$$

If we further resolve b_{11} into force-free and pressure gradient dependent contributions:
$$b_{11} = \tilde{b}_{11} + \overline{b}_{11}\,,$$
where $\overline{b}_{11}$ satisfies the equation
$$-2\,c_0\,\overline{b}_{11,l}|_\varphi + (b_{0,\varphi} + 3\,c_0')\,\overline{b}_{11} - 2\,b_0\,\overline{b}_{11,\varphi}$$
$$= \tfrac{3}{2}\,\varkappa\,p\,(2\,V_2 \sin\varphi + \cos\varphi\,V_{2,\varphi})\,, \qquad \text{(A 22)}$$
then the equation for $\tilde{b}_{11}$ is solved by
$$\tilde{b}_{11} = \tilde{b}_{11c}\cos u + \tilde{b}_{11s}\sin u\,, \qquad \text{(A 23)}$$
$$\tilde{b}_{11c} = \tfrac{1}{16}\,\{2\,\varkappa \cos a\,[12\,\tau\,c_0 + c_0\,(-5\,a' + 3\,K_0'\,e + 2\,K_0'/e)] + \sin a\,(c_0\,e'\,\varkappa/e - 5\,c_0'\,\varkappa - 2\,c_0\,\varkappa')\}\,,$$
$$\tilde{b}_{11s} = \tfrac{1}{16}\,\{2\,\varkappa \sin a\,[12\,\tau\,c_0 + c_0\,(-5\,a' + 2\,K_0'\,e + 3\,K_0'/e)] + \cos a\,(c_0\,e'\,\varkappa/e + 5\,c_0'\,\varkappa + 2\,c_0\,\varkappa')\}\,.$$

Introducing
$$\overline{b}_{11} = \overline{b}_{11c}\cos u + \overline{b}_{11s}\sin u\,,$$
with
$$\overline{b}_{11c} = \tfrac{3}{2}\,p\,\pi\,I\,c_0^{3/2}\,e^{+1/2}\,(b_i \cos a - b_r \sin a)\,,$$
$$\overline{b}_{11s} = \tfrac{3}{2}\,p\,\pi\,I\,c_0^{3/2}\,e^{-1/2}\,(b_r \cos a + b_i \sin a)\,, \qquad \text{(A 24)}$$

Equation (A 22) can be further reduced to an equation for the complex variable $b = b_r + i\,b_i$ reading
$$b' + i(K_0' - a')\,b \qquad \text{(A 25)}$$
$$= -\exp(i\,a)\,c_0^{-3/2}\,\varkappa\,(e^{-1/2}\cos a - i\,e^{1/2}\sin a)$$
with boundary conditions $b(L) = b(0)$. For the sake of brevity, we have not indicated here the way to obtain the solutions (A 23), (A 25) of Equation (A 20). One possible way is the theory of magnetic differential equations near the magnetic axis (see, for example [14]). Another way may be found within the framework of the Hamada formalism and is explained in more detail in Appendix II.

We now consider Equation (A 12). Introducing the Fourier analyzed form of V_3:
$$V_3 = V_{31c}\cos u + V_{31s}\sin u \qquad \text{(A 26)}$$
$$+ V_{33c}\cos 3\,u + V_{33s}\sin 3\,u$$
and substituting Eqs. (A 19), (21), (26) in Eq. (A 12) the latter one is resolved into four Fourier components analogous to Equation (A 26):
$$\begin{aligned} &-\tfrac{3}{2}\,c_0'\,V_{31c} - 2\,b_{0s}\,V_{31c} - 3\,b_{0s}\,V_{33c} + 2\,b_{0c}\,V_{31s} \\ &+ 3\,b_{0c}\,V_{33s} + 2\,V_{20}\,I_c - \tfrac{2}{3}\,V_{20}\,b_{11s} \\ &+ V_{2c}\,I_c - \tfrac{4}{3}\,V_{2c}\,b_{11s} - 2\,V_{2c}\,b_{13s} + b_{00}\,V_{31s} + c_0\,V_{31c}' \\ &+ c_0\,a'\,V_{31s} + 2\,c_0\,\varkappa\,V_{20}'\cos a \\ &+ c_0\,\varkappa\,V_{2c}'\cos a - 2\,c_0\,\varkappa\,a'\,V_{2c}\sin a = 0\,, \end{aligned} \qquad \text{(A 27a)}$$

$$-\tfrac{3}{2}c_0' V_{31s} + 2 b_{0s} V_{31s} - 3 b_{0s} V_{33s} + 2 b_{0c} V_{31c} - 3 b_{0c} V_{33c} + 2 V_{20} I_s + \tfrac{2}{3} V_{20} b_{11c} - V_{2c} I_s - \tfrac{4}{3} V_{2c} b_{11c} + 2 V_{2c} b_{13c} - b_{00} V_{31c} + c_0 V_{31s}' - c_0 a' V_{31c} + 2 c_0 \varkappa V_{20}' \sin\alpha - c_0 \varkappa V_{2c}' \sin\alpha - 2 c_0 \varkappa a' V_{2c} \cos\alpha = 0, \quad \text{(A 27b)}$$

$$-b_{0s} V_{31c} - b_{0c} V_{31s} + V_{2c} I_c + \tfrac{2}{3} V_{2c} b_{11s} + c_0 \varkappa V_{2c}' \cos\alpha + 2 c_0 \varkappa a' V_{2c} \sin\alpha - \tfrac{3}{2} c_0' V_{33c} - 2 V_{20} b_{13s} + 3 b_{00} V_{33s} + c_0 V_{33c}' + 3 c_0 a' V_{33s} = 0, \quad \text{(A 27c)}$$

$$-b_{0s} V_{31s} + b_{0c} V_{31c} + V_{2c} I_s - \tfrac{2}{3} V_{2c} b_{11c} + c_0 \varkappa V_{2c}' \sin\alpha - 2 c_0 \varkappa a' V_{2c} \cos\alpha - \tfrac{3}{2} c_0' V_{33s} - 2 V_{20} b_{13c} - 3 b_{00} V_{33c} + c_0 V_{33s}' - 3 c_0 a' V_{33c} = 0. \quad \text{(A 27d)}$$

The equations contain the quantities b_{13c} and b_{13s}, which are still unknown. Since we want to prescribe geometrical quantities in third order rather than the Fourier components b_{13c}, b_{13s} of the third-order transverse magnetic field, we eliminate b_{13c} and b_{13s}. The resulting two equations contain only the Fourier components of V_3 as unknowns. We now use

$$V_{31c} = \pi^2 L^{-1} c_0 I \left[\left(3e + \frac{1}{e}\right)\overline{S}_c - \delta\right],$$
$$V_{31s} = \pi^2 L^{-1} c_0 I \left[\left(\frac{1}{e} + \frac{3}{e^3}\right)\overline{S}_s - \Delta\right],$$
$$V_{33c} = \pi^2 L^{-1} c_0 I \left[\left(e - \frac{1}{e}\right)\overline{S}_c + \delta\right], \quad \text{(A 28)}$$
$$V_{33s} = \pi^2 L^{-1} c_0 I \left[\left(\frac{1}{e} - \frac{3}{e^3}\right)\overline{S}_s - \Delta\right],$$

where

$$S\cos\alpha - s\sin\alpha = \overline{S}_c, \quad S\sin\alpha + s\cos\alpha = \overline{S}_s \quad \text{(A 29)}$$

and which is obtained from Equation (5). Finally, we substitute Eqs. (A 13), (15), (16), (19), (23), (28) in the two equations for the Fourier components of V_3 and get

$$\left(e + \frac{3}{e}\right)\left[-\frac{1}{2}\left(\frac{c_0'}{c_0} + \frac{e'}{e}\right)\overline{S}_c + \overline{S}_c' + \frac{1}{e}K_0'\overline{S}_s\right] + \frac{1}{2}\left(\frac{c_0'}{c_0} - \frac{e'}{e}\right)\delta - \delta' - e K_0' \Delta = R_1,$$

$$\left(3e + \frac{1}{e}\right)\left[-\left(\frac{1}{2}\frac{c_0'}{c_0} + \frac{3}{2}\frac{e'}{e}\right)\overline{S}_s + \overline{S}_s' - e K_0' \overline{S}_c\right] - e^2\left[-\frac{1}{2}\left(\frac{c_0'}{c_0} + \frac{e'}{e}\right)\Delta + \Delta'\right] + e K_0' \delta = R_2,$$

$$R_1 = \frac{L\varkappa}{8\pi}\left\{2\sin\alpha\left[-K_0' + a'\left(3e - \frac{2}{e}\right) + 4\tau e\right] + \cos\alpha\left[\frac{c_0'}{c_0}\left(e - \frac{2}{e}\right) + 2\frac{\varkappa'}{\varkappa}\left(e + \frac{2}{e}\right) - \frac{e'}{e}\left(3e + \frac{2}{e}\right)\right] + \frac{16}{3}\frac{e}{\varkappa c_0}\overline{b}_{11s}\right\},$$

$$R_2 = e^2 \frac{L\varkappa}{8\pi}\left\{2\cos\alpha\left[K_0' + a'\left(2e - \frac{3}{e}\right) - 4\tau/e\right] + \sin\alpha\left[\frac{c_0'}{c_0}\left(\frac{1}{e} - 2e\right) + 2\frac{\varkappa'}{\varkappa}\left(2e + \frac{1}{e}\right) + \frac{e'}{e}\left(2e + \frac{3}{e}\right)\right] - \frac{16}{3}\frac{1}{e c_0 \varkappa}\overline{b}_{11c}\right\}. \quad \text{(A 30)}$$

If c_0 and e are not constant, it is useful to reduce these equations further by introducing

$$S_c^* = c_0^{-1/2} e^{-1/2} \overline{S}_c, \quad S_s^* = c_0^{-1/2} e^{-3/2} \overline{S}_s,$$
$$\delta^* = c_0^{-1/2} e^{1/2} \delta, \quad \Delta^* = c_0^{-1/2} e^{-1/2} \Delta.$$

Equations (A 30) then read

$$\left(e + \frac{3}{e}\right)[S_c^{*\prime} + K_0' S_s^*] - \frac{1}{e}\delta^{*\prime} - e K_0' \Delta^* = c_0^{-1/2} e^{-1/2} R_1, \quad \text{(A 31)}$$

$$\left(3e + \frac{1}{e}\right)[S_s^{*\prime} - K_0' S_c^*] - e\Delta^{*\prime} + \frac{1}{e}K_0'\delta^* = c_0^{-1/2} e^{-3/2} R_2.$$

Here, in general, two of the four quantities S_c^*, S_s^*, δ^*, and Δ^* may be prescribed. The calculation of third-order quantities can be completed by obtaining b_{13s} and b_{13c} from Eqs. (A 27 c, d) and c_3 from Equation (A 8).

Appendix II

Here we develop the Hamada formalism as far as is necessary for the evaluation of the criteria (18) – (20). We have to solve the equilibrium equations written in Hamada coordinates:

$$\mathbf{B} = \dot{\chi}\,\boldsymbol{r}_{,\Theta} + \dot{\Phi}\,\boldsymbol{r}_{,\zeta}, \quad \iota = \dot{\chi}/\dot{\Phi},$$
$$\boldsymbol{j} = \dot{I}\,\boldsymbol{r}_{,\Theta} + \dot{J}\,\boldsymbol{r}_{,\zeta}, \quad \text{(B 1a – g)}$$
$$\dot{p} = \dot{I}\dot{\Phi} - \dot{J}\dot{\chi}, \quad \boldsymbol{r}_{,V} \cdot (\boldsymbol{r}_{,\Theta} \times \boldsymbol{r}_{,\zeta}) = 1,$$

$$(\dot{\Phi} g_{\zeta\zeta} + \dot{\chi} g_{\Theta\zeta})_{,\Theta} - (\dot{\Phi} g_{\Theta\zeta} + \dot{\chi} g_{\Theta\Theta})_{,\zeta} = 0,$$
$$(\dot{\Phi} g_{V\zeta} + \dot{\chi} g_{V\Theta})_{,\zeta} - (\dot{\Phi} g_{\zeta\zeta} + \dot{\chi} g_{\Theta\zeta})_{,V} = \dot{I},$$
$$(\dot{\Phi} g_{\Theta\zeta} + \dot{\chi} g_{\Theta\Theta})_{,V} - (\dot{\Phi} g_{V\zeta} + \dot{\chi} g_{V\Theta})_{,\Theta} = \dot{J}$$

in the neighbourhood of the magnetic axis by an expansion of the position vector $\boldsymbol{r}(V, \Theta, \zeta)$ in powers of $V^{1/2}$. In order to be able to use the Frenet formulae

$$\boldsymbol{t}' = \varkappa \boldsymbol{n}, \quad \boldsymbol{n}' = -(\tau \boldsymbol{b} + \varkappa \boldsymbol{t}), \quad \boldsymbol{b}' = \tau \boldsymbol{n}, \quad \text{(B 2)}$$

where the prime indicates the derivative with respect to the arc length l of the magnetic axis, we perform the expansion in two stages

$$\boldsymbol{r}(V, \Theta, l) = \boldsymbol{r}_0(l) + \boldsymbol{r}_1(l, \Theta) V^{1/2} + \boldsymbol{r}_2(l, \Theta) V + O(V^{3/2}),$$
$$l(V, \Theta, \zeta) = l_0(\zeta) + l_1(\Theta, \zeta) V^{1/2} + l_2(\Theta, \zeta) V + O(V^{3/2}). \quad \text{(B 3)}$$

Here, $\boldsymbol{r}_0(l)$ describes the magnetic axis and

$$\boldsymbol{r}_1 = \xi(l, \Theta) \boldsymbol{n} + \eta(l, \Theta) \boldsymbol{b}, \quad \boldsymbol{r}_2 = \xi_2 \boldsymbol{n} + \eta_2 \boldsymbol{b} \quad \text{(B 4)}$$

lie in the plane perpendicular to the magnetic axis at the value l of the arc length. One then obtains

$$\boldsymbol{r}_{,\Theta} = (\boldsymbol{t}\, l_{1,\Theta} + \boldsymbol{r}_{1,\Theta}) V^{1/2} + (\boldsymbol{t}\, l_{2,\Theta} + \boldsymbol{r}_{1,l}\, l_{1,\Theta} + \boldsymbol{r}_{2,\Theta}) V + O(V^{3/2}),$$
$$\boldsymbol{r}_{,\zeta} = \boldsymbol{t}\, l_{0,\zeta} + (\boldsymbol{t}\, l_{1,\zeta} + \boldsymbol{r}_{1,l}\, l_{0,\zeta}) V^{1/2} + O(V),$$
$$\boldsymbol{r}_{,V} = \tfrac{1}{2}(\boldsymbol{t}\, l_1 + \boldsymbol{r}_1) V^{-1/2} + O(V^0),$$

so that

$$g_{\Theta\Theta} = (l_{1,\Theta}{}^2 + \xi_{,\Theta}{}^2 + \eta_{,\Theta}{}^2) V + O(V^{3/2}),$$
$$g_{\zeta\zeta} = l_{0,\zeta}{}^2 + 2\, l_{0,\zeta}(l_{1,\zeta} - \varkappa\, l_{0,\zeta}\, \xi) V^{1/2} + O(V),$$
$$g_{VV} = \tfrac{1}{4}(l_1{}^2 + \xi^2 + \eta^2) V^{-1} + O(V^{-1/2}),$$
$$g_{\Theta\zeta} = l_{0,\zeta}\, l_{1,\Theta} V^{1/2} + [l_{0,\zeta}\, l_{2,\Theta} - 2\, l_{1,\Theta}\, l_{0,\zeta}\, \varkappa\, \xi + l_{1,\zeta}\, l_{1,\Theta} + l_{0,\zeta}\, \tau(\xi_{,\Theta}\, \eta - \eta_{,\Theta}\, \xi) + l_{0,\zeta}(\xi_{,\Theta}\, \xi_{,l} + \eta_{,\Theta}\, \eta_{,l})] V + O(V^{3/2}),$$
$$g_{\Theta V} = \tfrac{1}{2}(l_1\, l_{1,\Theta} + \xi\, \xi_{,\Theta} + \eta\, \eta_{,\Theta}) + O(V^{1/2}),$$
$$g_{\zeta V} = \tfrac{1}{2}\, l_{0,\zeta}\, l_1\, V^{-1/2} + O(V^0), \quad \text{(B 5)}$$

We now establish the relationship of the Hamada coordinates V, Θ, ζ to the geometrical ϱ, φ, l. Because of Eq. (B 1a)

$$l_{0,\zeta} = c_0 / \dot{\Phi}_0,$$

so that

$$\dot{\Phi}_0 = I^{-1}, \quad I = \oint \mathrm{d}l / c_0$$
$$\zeta = I^{-1} \int_0^l \mathrm{d}l / c_0 + O(V^{1/2}). \quad \text{(B 6)}$$

Because of Eq. (B 3), (B 4)

$$\varrho \cos\varphi = \xi V^{1/2} + O(V),$$
$$\varrho \sin\varphi = \eta V^{1/2} + O(V),$$

which leads with the geometrical representation of the volume

$$V = \pi\, c_0\, I\, \varrho^2 [e \cos^2 u + (1/e) \sin^2 u] + O(\varrho^3) \quad \text{(B 7)}$$

to $(\varphi = u - \alpha)$

$$\xi = (\pi\, l_{0,\zeta})^{-1/2} [e \cos^2 u + (1/e) \sin^2 u]^{-1/2} \cdot (\cos u \cos\alpha + \sin u \sin\alpha),$$
$$\eta = (\pi\, l_{0,\zeta})^{-1/2} [e \cos^2 u + (1/e) \sin^2 u]^{-1/2} \cdot (\sin u \cos\alpha - \cos u \sin\alpha).$$

Since

$$\cos u\, [e \cos^2 u + (1/e) \sin^2 u]^{-1/2} = e^{-1/2} \cos[\arctan(e^{-1} \tan u)],$$
$$\sin u\, [e \cos^2 u + (1/e) \sin^2 u]^{-1/2} = e^{1/2} \sin[\arctan(e^{-1} \tan u)],$$

we choose for the Hamada coordinate Θ

$$2\pi\, \Theta = \arctan(e^{-1} \tan u) - K(l) + O(V^{1/2}), \quad \text{(B 8)}$$

where $K(l)$ is arbitrary at this point except for the periodicity condition

$$K(L) - K(O) = \oint \tau\, \mathrm{d}l + \alpha(L) - \alpha(0),$$

where $\alpha(L) - \alpha(O) = n\pi$ ($n/2$ is the number of full turns of the elliptical cross-section) and $\oint \tau\, \mathrm{d}l = 2\pi m$ (m is the number of turns of the normal over the length L of the magnetic axis). One then obtains

$$\xi = \xi_c \cos 2\pi\Theta + \xi_s \sin 2\pi\Theta, \quad \text{(B 9)}$$
$$\eta = \eta_c \cos 2\pi\Theta + \eta_s \sin 2\pi\Theta,$$
$$\xi_c = (\pi\, l_{0,\zeta})^{-1/2} (e^{1/2} \sin K \sin\alpha + e^{-1/2} \cos K \cos\alpha),$$
$$\xi_s = (\pi\, l_{0,\zeta})^{-1/2} (e^{1/2} \cos K \sin\alpha - e^{-1/2} \sin K \cos\alpha),$$
$$\eta_c = (\pi\, l_{0,\zeta})^{-1/2} (e^{1/2} \sin K \cos\alpha - e^{-1/2} \cos K \sin\alpha),$$
$$\eta_s = (\pi\, l_{0,\zeta})^{-1/2} (e^{1/2} \cos K \cos\alpha + e^{-1/2} \sin K \sin\alpha).$$

We now solve the equilibrium Eqs. (B 1 d – g):

$O(V^0)$ of Eq. (B 1 d):

$$1 = \tfrac{1}{2}\, l_{0,\zeta}(\xi\, \eta_{,\Theta} - \eta\, \xi_{,\Theta}) = \pi\, l_{0,\zeta}(\xi_c\, \eta_s - \xi_s\, \eta_c) \quad \text{(B 10)}$$

$O(V^{1/2})$ of Eq. (B 1 e):

$$l_{0,\zeta}\, l_{1,\Theta\zeta} - l_{0,\zeta\zeta}\, l_{1,\Theta} + \iota\, l_{0,\zeta}\, l_{1,\Theta\Theta} = 2\varkappa\, l_{0,\zeta}{}^2\, \xi_{,\Theta},$$

$O(V^{-1/2})$ of Eq. (B 1 f):

$$l_{0,\zeta}\, l_{1,\zeta} - l_{0,\zeta\zeta}\, l_1 + \iota\, l_{0,\zeta}\, l_{1,\Theta} = 2\varkappa\, l_{0,\zeta}{}^2\, \xi, \quad \text{(B 11)}$$

$O(V^{-1/2})$ of Eq. (B 1 g):

$$\tfrac{1}{2}\, l_{0,\zeta}\, l_{1,\Theta} - \tfrac{1}{2}\, l_{1,\Theta}\, l_{0,\zeta} = 0,$$

$O(V^0)$ of Eq. (B 1 g); mean value with respect to Θ of this equation:

$$-2\, l_{0,\zeta}\, \varkappa \langle l_{1,\Theta}\, \xi \rangle + \langle l_{1,\zeta}\, l_{1,\Theta} \rangle + 2\, l_{0,\zeta}\, \tau \langle \xi_{,\Theta}\, \eta \rangle + l_{0,\zeta} \langle \xi_{,\Theta}\, \xi_{,l} + \eta_{,\Theta}\, \eta_{,l} \rangle + \iota \langle l_{1,\Theta}{}^2 + \xi_{,\Theta}{}^2 + \eta_{,\Theta}{}^2 \rangle = \dot{J}/\dot{\Phi}. \quad \text{(B 12)}$$

Equations (B 9) solve Eq. (B 10), and Eq. (B 11) is further reduced by Fourier analysis of l_1:

$$l_1 = l_c \cos 2\pi\Theta + l_s \sin 2\pi\Theta \qquad \text{(B 13)}$$

$$l_{c,\zeta} - (l_{0,\zeta\zeta}/l_{0,\zeta})\, l_c + 2\pi\iota\, l_s = 2\varkappa\, l_{0,\zeta}\, \xi_c ,$$

$$l_{s,\zeta} - (l_{0,\zeta\zeta}/l_{0,\zeta})\, l_s - 2\pi\iota\, l_c = 2\varkappa\, l_{0,\zeta}\, \xi_s .$$

Using Eqs. (B 9), (11), one can finally reduce Eq. (B 12) to

$$K'(l) = \frac{1}{e+1/e}\,(\dot J/\dot\Phi + 2\tau + 2\alpha') - \frac{2\pi\iota}{c_0 I} = K_0' - 2\pi\iota/c_0 I \qquad \text{(B 14)}$$

[see Equation (10)].

For the evaluation of Eqs. (18) – (20) we need $|\nabla\zeta|^2$ and $g_{\Theta\Theta}|\nabla V|^2$ on the magnetic axis:

$$|\nabla\zeta|^2 = g_{VV}\, g_{\Theta\Theta} - g_{V\Theta}{}^2 = \frac{1}{l_{0,\zeta}{}^2} + \pi^2[(\eta_s l_c - \eta_c l_s)^2 + (\xi_s l_c - \xi_c l_s)^2] ,$$

$$g_{\Theta\Theta}|\nabla V|^{-2} = \frac{1}{g_{\zeta\zeta} - g_{\Theta\zeta}{}^2/g_{\Theta\Theta}} = \frac{1}{l_{0,\zeta}{}^2}\left(1 + \frac{l_{1,\Theta}{}^2}{\xi_{,\Theta}{}^2 + \eta_{,\Theta}{}^2}\right). \qquad \text{(B 15)}$$

Integrating over Θ one finally gets

$$\int g_{\Theta\Theta}|\nabla V|^{-2}\, d\Theta = \frac{1}{l_{0,\zeta}{}^2}\left\{1 + \frac{\pi l_{0,\zeta}}{2}\left[l_c^2 + l_s^2 - \frac{(l_s^2 - l_c^2)(\eta_s^2 - \eta_c^2 + \xi_s^2 - \xi_c^2) + 4 l_c l_s(\eta_c\eta_s + \xi_c\xi_s)}{\dfrac{2}{\pi l_{0,\zeta}} + \eta_c^2 + \eta_s^2 + \xi_c^2 + \xi_s^2}\right]\right\}. \qquad \text{(B 16)}$$

Appendix III

Here, we prove the sufficient stability criterion Eq. (19) by evaluating the criterion obtained in [10] near the magnetic axis. In terms of

$$A = \frac{2}{|\nabla V|^6}(\boldsymbol{j}\times\nabla V)\cdot(\boldsymbol{B}\cdot\nabla)\nabla V = \frac{1}{|\nabla V|^2} \cdot\left\{\frac{\boldsymbol{j}^2}{|\nabla V|^2} + \dot I\ddot\Phi - \dot J\ddot\chi - (\boldsymbol{B}\cdot\nabla)(\dot I g^{V\zeta} - \dot J g^{V\Theta})\frac{1}{|\nabla V|^2}\right\}$$

(see, for example [15]), the latter criterion reads: Stability holds if a single-valued function Λ exists which satisfies the inequality

$$\boldsymbol{B}\cdot\nabla\Lambda - |\nabla V|^2\Lambda^2 - |\nabla V|^2 A \geqq 0 . \qquad \text{(C 1)}$$

For $\dot J = 0$ (which is the only case in which the criterion can be satisfied) we obtain

$$|\nabla V|^2 A = \dot I^2 g_{\Theta\Theta}/|\nabla V|^2 + \dot I\ddot\Phi - (\boldsymbol{B}\cdot\nabla)\, a \qquad \text{(C 2)}$$

$$= -\left(\dot\Phi_0\frac{\partial}{\partial\zeta} + \dot\chi_0\frac{\partial}{\partial\Theta}\right) a_{-1} V^{-1/2} + O(V^0) ,$$

where

$$a = \frac{\dot I g^{V\zeta}}{|\nabla V|^2} = a_{-1} V^{-1/2} + O(V^0) . \qquad \text{(C 3)}$$

Expanding Eq. (C 1) we therefore find

$$\Lambda = \Lambda_{-1} V^{-1/2} + \Lambda_0 + O(V^{1/2}) ,$$

with

$$\Lambda_{-1} = -a_{-1} , \qquad \text{(C 4)}$$

$$\left(\dot\Phi_0\frac{\partial}{\partial\zeta} + \dot\chi_0\frac{\partial}{\partial\Theta}\right)\Lambda_0 - |\nabla V|^2\Lambda_{-1}{}^2 V^{-1} - (|\nabla V|^2 A)_0 \geqq 0 . \qquad \text{(C 5)}$$

The solubility condition for Eq. (C 5) is

$$\oint\frac{dl}{c_0}\,(V^{-1}|\nabla V|^2\Lambda_{-1}{}^2 + |\nabla V|^2 A) \leqq 0 ,$$

which is the stability criterion near the magnetic axis. (A similar argument was used in [16] to obtain a sufficient stability criterion without expansion near the magnetic axis if A is expandable in a small parameter.) Using Eqs. (C 2, 3, 4), one finally obtains (for $\dot p < 0$)

$$\ddot\Phi\,\dot\Phi + \dot p\oint|\nabla\zeta|^2\, d\zeta > 0 .$$

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